ON THE STABILITY OF THE STATIONARY MOTION OF THE GYROSCOPIC FRAME

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A rigid body, which we shall call the frame, can perform all possible rotations about the fixed point O inside the frame (Fig. 1). The frame contains two gyroscopes whose inner rings [housings] could rotate about parallel axes fixed in the frame, through the same rotation angle δ. Further, the inner rings of the two gyroscopes act on each other through some mechanism, such as a spring. The center of gravity of the whole system does not coincide with the point O. This system thus resembles the gyrosphere of a space compass.

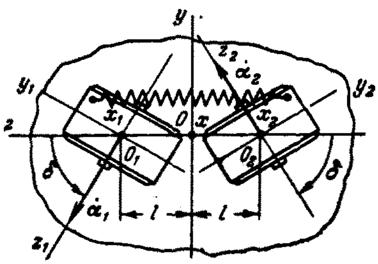

Fig. 1.

Ishlinskii [1,2] investigated a similar system with a moving base, and using the elementary gyroscope theory, has shown many of its interesting properties. It serves a useful purpose to examine rigorously some of these properties. In addition, Rumiantsev has successfully carried out rigorous investigations of the dynamics of a rigid body with one point fixed. In particular, he skillfully utilized the Routh-Liapunov theorem on the stability of stationary motion when he investigated the stability of permanent rotations of a heavy rigid body [3].

This paper considers the problem of the existence of stationary motions and their stability. The mass of the frame is taken into account. The gyroscopes are not the "fast-spinning" ones as it is customary to assume in elementary theory. In general, the spin velocities of the two gyroscopes are different. The only external force is the force of gravity. It is assumed that the system is conservative and that the frame of reference is inertial.

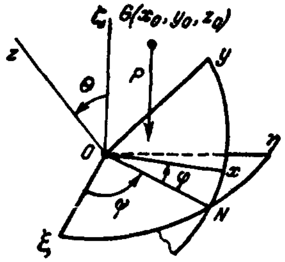

Fig. 2.

1. Let $O\xi\eta\zeta$ be an inertial coordinate system (Fig. 2), the coordinate system $Oxyz$ be fixed in the frame, its axes being the principal axes of inertia of the frame through the point O (Figs. 1, 2), the x-axis parallel to the rotation axes of the inner rings, the z_1- and z_2-axes be the spin axes of the gyroscopes located in the Oyz plane. Then the position of the whole system with respect to the axes $O\xi\eta\zeta$ can be determined through the angles ϕ, ψ, θ, a_1 and a_2. Here a_1 and a_2 are the rotation angles of the gyroscopes with respect to their inner rings.

The mass of each gyroscope is m, its equatorial (for the axes through O_1 and O_2) and axial moments of inertia are A_1 and C_1, respectively; the moments of inertia of the frame about the axes x, y and z are A_2, B_2 and C_2, respectively.

Let also p, q and r be the x, y and z components of the angular velocity:

$$p = \dot\psi \sin\theta \sin\varphi + \theta \cos\varphi, \qquad q = \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi, \qquad r = \dot\psi \cos\theta + \dot\varphi$$

The z_1 and z_2 components of the absolute angular velocity of each gyroscope are

$$r_1 = \dot\alpha_1 - q \sin\delta + r\cos\delta, \qquad r_2 = \dot\alpha_2 + q\sin\delta + r\cos\delta$$

The kinetic energy of the system is

$$2T = Ap^2 + Bq^2 + Cr^2 + 2A_1\dot\delta^2 + C_1(r_1^2 + r_2^2) \tag{1.1}$$

where

$$A = A_2 + 2ml^2 + 2A_1, \quad B = B_2 + 2ml^2 + 2A_1\cos^2\delta, \quad C = C_2 + 2A_1\sin^2\delta$$

The expression for the force function has the form

$$U = -P(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) + 2\int M(\delta)\,d\delta \tag{1.2}$$

Here P is the weight of the system, $M(\delta)$ is the moment of the spring system, x_0, y_0 and z_0 are the coordinates of the center of gravity with respect to $Oxyz$. The direction cosines γ_1, γ_2 and γ_3, of the upward-directed vertical axis ζ, with respect to the x-, y- and z-axes are

$$\gamma_1 = \sin\theta\sin\varphi, \qquad \gamma_2 = \sin\theta\cos\varphi, \qquad \gamma_3 = \cos\theta$$

From (1.1) and (1.2) it follows that the coordinates ψ, a_1 and a_2, are cyclic. The first integrals of the equations of motion with respect to these coordinates are

$$C_1r_1 = H_1 = \text{const}, \qquad C_1r_2 = H_2 = \text{const} \qquad (H = H_1 + H_2,\ h = H_2 - H_1) \qquad (1.3)$$
$$Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + h\gamma_2 \sin\delta + H\gamma_3 \cos\delta = n = \text{const}$$

We shall eliminate the cyclic coordinates ψ, a_1 and a_2. After certain transformations the Routh function assumes the form

$$R = \frac{1}{2}\left[\left(I_3 - \frac{I_2^2}{I_1}\right)\dot\theta^2 - 2\frac{I_2}{I_1}C\gamma_3\dot\theta\dot\varphi + \left(C - \frac{C^2}{I_1}\gamma_2^3\right)\dot\varphi^2 + 2A_1\dot\delta^2\right] + \left[H\cos\delta + \right.$$
$$\left. + \frac{C}{I_1}(n - h\gamma_2\sin\delta - H\gamma_3\cos\delta)\gamma_3\right]\dot\varphi + \left[-h\sin\varphi\sin\delta + \frac{I_2}{I_1}(n - h\gamma_2\sin\delta - \right.$$
$$\left. - H\gamma_3\cos\delta)\right]\dot\theta - \frac{1}{2I_1}(n - h\gamma_2\sin\delta - H\gamma_3\cos\delta)^2 \qquad (1.4)$$

Here

$$I_1 = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2, \qquad I_2 = A\gamma_1\cos\varphi - B\gamma_2\sin\varphi, \qquad I_3 = A\cos^2\varphi + B\sin^2$$

For the coordinates $q_1 = \theta$, $q_2 = \phi$, $q_3 = \delta$, we have the equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q}\right) - \frac{\partial R}{\partial q} = \frac{\partial U}{\partial q} \qquad (1.5)$$

We shall consider the stationary motion of the system

$$\theta = \theta_0, \quad \varphi = \varphi_0, \quad \delta = \delta_0, \quad \dot\alpha_1 = \omega_1, \quad \dot\alpha_2 = \omega_2, \quad \dot\psi = \omega \qquad (1.6)$$

which corresponds to the uniform rotation of the frame about a vertical axis. The constants (1.6) should satisfy the condition $\partial/\partial q\ (R - U) = 0$. Replacing the symbols γ_1, γ_2 and γ_3, by α, β and γ, respectively, and taking into account on the strength of (1.3) that

$$\omega = \frac{1}{I_{10}}(n - h\beta\sin\delta_0 - H\gamma\cos\delta_0)$$

we obtain the following conditions:

$$(H\sin\theta_0\cos\delta_0 - h\gamma\cos\varphi_0\sin\delta_0)\,\omega - (A\alpha\sin\varphi_0 + B_0\beta\cos\varphi_0 - C_0\sin\theta_0)\,\gamma\omega^2 =$$
$$= -P\,(x_0\gamma\sin\varphi_0 + y_0\gamma\cos\varphi_0 - z_0\sin\theta_0)$$
$$h\omega\alpha\sin\delta_0 - (A - B_0)\,\omega^2\alpha\beta = -P\,(x_0\beta - y_0\alpha) \qquad (1.7)$$
$$(H\gamma\sin\delta_0 - h\beta\cos\delta_0)\omega - A_1(\gamma^2 - \beta^2)\,\omega^2\sin 2\delta_0 = 2M(\delta_0)$$

Here I_{10}, B_0, C_0 denote the values of these quantities for the motion under consideration. The last equation in (1.7) determines the moment of the spring system. When θ_0, ϕ_0, δ_0 and ω are assigned, then the quantities H and h can be determined from the first two equations. Rotations about the principal axes of inertia of the frame are investigated in [6].

Consequently, by utilizing gyroscopes we can make the frame rotate about an arbitrary vertical axis by selecting appropriate spin velocities for the gyroscopes. (If $a = 0$, then at $\beta \neq 0$ it follows that $x_0 = 0$).

We shall separate now the rotations of the frame about axes lying in the principal planes of the frame, by setting $y_0 = z_0 = 0$. When the axes are in the Oxy plane ($\gamma = 0$) the conditions (1.7) give $H = 0$, which is equivalent to $\omega_1 = -\omega_2$.

$$\begin{gathered} h\omega\alpha \sin\delta_0 - (A - B_0)\,\omega^2\alpha\beta = -Px_0\beta, \\ -h\omega\beta\cos\delta_0 + A_1\beta^2\omega^2 \sin 2\delta_0 = 2M(\delta_0), \qquad h = 2C_1(\omega_2 + \omega\beta\sin\delta_0) \end{gathered} \tag{1.8}$$

When the axes are in the Oxz plane ($\beta = 0$), the conditions are analogous to (1.8) where the quantities h, δ_0, B_0 and β are replaced by H, $1/2\pi + \delta_0$, C_0 and γ, respectively.

Rotations about the axes which are in the Oyz plane ($a = 0$) are impossible when $x_0 = 0$.

2. We shall investigate the stability of rotation about the axes which are lying in the Oxy plane, that is, we shall set in the perturbed motion $\theta = 1/2\,\pi + x_1$, $\phi = \phi_0 + x_2$, $\delta = \delta_0 + x_3$.

The variational equations of the system (1.5) will have the following form:

$$\frac{AB_0}{I_{10}}\ddot{x}_1 - m\dot{x}_2 + k\dot{x}_3 + a_1x_1 = 0, \quad \begin{matrix} C_0\ddot{x}_2 + m\dot{x}_1 + a_2x_2 + bx_3 = 0 \\ 2A_1\ddot{x}_3 - k\dot{x}_1 + bx_2 + a_3x_3 = 0 \end{matrix} \tag{2.1}$$

We have the following expressions for the coefficients of the gyroscopic forces:

$$\begin{gathered} m = \frac{1}{I_{10}}\,[B_0h\beta\sin\delta_0 + (A - B_0)(A\alpha^2 - B_0\beta^2)\,\omega - I_{10}C_0\omega] \\ k = \frac{A\alpha}{I_{10}}(2A_1\omega\beta\sin 2\delta_0 - h\cos\delta_0) \end{gathered} \tag{2.2}$$

In the coefficients of the potential forces we can eliminate Px_0 by the use of (1.8) and then put them in the form

$$a_1 = \frac{1}{\beta}\,h\omega\sin\delta_0 + (B_0 - C_0)\,\omega^2, \quad a_2 = \frac{1}{I_{10}}\Big\{h^2\alpha^2\sin^2\delta_0 + \left[\frac{I_{10}}{\beta^2} - 4(A - B_0)\,\alpha^2\right]h\omega\beta\sin\delta_0 + \left[3(A - B_0)\,\alpha^2 - B_0\right](A - B_0)\,\omega^2\beta^2\Big\} \tag{2.3}$$

$$a_3 = \frac{1}{I_{10}}\Big[h^2\beta^2\cos^2\delta_0 + (I_{10} - 8A_1\beta^2\cos^2\delta_0)\,h\omega\beta\sin\delta_0 + \\ + (I_{10}\cos 2\delta_0 + 2A_1\beta^2\sin^2 2\delta_0)\,2A_1\omega^2\beta^2\Big] - 2\left(\frac{dM(\delta)}{d\delta}\right)_{\delta=\delta}$$

$$b = -\frac{1}{I_{10}}\Big\{\frac{1}{2}h^2\alpha\beta\sin 2\delta_0 - [I_{10} + 2(A - B_0)\beta^2 + 4A_1\beta^2\sin^2\delta_0]\,h\omega\alpha\cos\delta_0 + \\ + [I_{10} + 2(A - B_0)\beta^2]\,2A_1\omega^2\alpha\beta\sin 2\delta_0\Big\}$$

It is known [4] that if the unperturbed motion is stable then Equations (2.1) permit a sign-definite quadratic integral, which is the energy integral, obtainable also for the equations of the perturbed motion. It has the form

$$V = \frac{AB_0}{I_{10}}\dot{x}_1^2 + C_0\dot{x}_2^2 + 2A_1\dot{x}_3^2 + a_1x_1^2 + a_2x_2^2 + 2bx_2x_3 + a_3x_3^2 + \ldots = \text{const} \qquad (2.4)$$

where $2U^* = -(a_1x_1^2 + a_2x_2^2 + 2bx_2x_3 + a_3x_3^2 + \ldots)$ is the variable force function. Consequently, the sufficient condition of stability for the considered stationary motion will be the condition for sign-definiteness (positive-definiteness) of the integral (2.4):

$$a_1 > 0, \qquad a_2 > 0, \qquad a_2a_3 - b^2 > 0 \qquad (2.5)$$

We note that if the gyroscopes are removed, that is, if we set $C_1 = A_1 = 0$, then the resulting conditions coincide with those obtained by Rumiantsev [3].

Equations (2.1) in normal coordinates have to be transformed accordingly. If, now, C_1, C_2 and C_3 are the Poincare coefficients of stability, then we have the well-known relations (see, for example, [5])

$$c_1c_2c_3 = \mu a_1(a_2a_3 - b^2) \qquad (\mu > 0) \qquad (2.6)$$

Thus, if $a_1(a_2a_3 - b^2) < 0$, then the degree of instability is odd, and on the strength of Kelvin's theorem [5] we conclude that the motion is unstable. If $a_1(a_2a_3 - b^2) > 0$ with the other conditions in (2.5) violated, there exists a possibility of a gyroscopic stabilization of the unstable equilibrium of the conservative system (2.1), and the problem remains open. When the axes are lying in the Ox -plane then the conditions of stability are similar to (2.5) with an appropriate change of symbols.

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